

Naturalness in Cosmological Initial Conditions

F. Nitti^a, M. Porrati^{a,b} and J.-W. Rombouts^a

^a *Department of Physics, NYU, 4 Washington Pl., New York, NY 10003, USA*

^b *Scuola Normale Superiore, Piazza dei Cavalieri 7, I-56126, Pisa, Italy*

ABSTRACT

We propose a novel approach to the problem of constraining cosmological initial conditions. Within the framework of effective field theory, we classify initial conditions in terms of boundary terms added to the effective action describing the cosmological evolution below Planckian energies. These boundary terms can be thought of as spacelike branes which may support extra instantaneous degrees of freedom and extra operators. Interactions and renormalization of these boundary terms allow us to apply to the boundary terms the field-theoretical requirement of naturalness, i.e. stability under radiative corrections. We apply this requirement to slow-roll inflation with non-adiabatic initial conditions, and to cyclic cosmology. This allows us to define in a precise sense when some of these models are fine-tuned. We also describe how to parametrize in a model-independent way non-Gaussian initial conditions; we show that in some cases they are both potentially observable and pass our naturalness requirement.

Contents

1	Introduction	2
2	EFT of Cosmological Initial Conditions	5
2.1	Inflation	5
2.2	Changing the Initial State	6
2.3	Power Spectrum	6
2.4	Scalar Metric Fluctuations	7
2.5	Cyclic Cosmologies	8
3	Naturalness	11
3.1	Naturalness in Inflation I	12
3.2	Naturalness in Inflation II	15
3.3	Naturalness in Inflation III	17
3.4	Coda: Is the BD Vacuum Natural?	19
4	Naturalness in Cyclic Cosmologies	20
4.1	Scale-Invariant Spectrum	20
4.2	Causality Concerns	23
4.3	Naturalness Concerns I	24
4.4	Naturalness Concerns II	24
5	Non-Gaussianities in Inflation	27
5.1	Minimally Coupled Scalar in de Sitter Space	27
5.1.1	Estimating the Back-Reaction	29
5.2	Scalar Fluctuations	30
6	Non-Gaussianities in Cyclic Cosmologies	31
6.1	Damping Effects	31
6.2	Non-Linear Interactions in the Slow-Contraction Epoch	31
7	Conclusions	32

*Pronaque cum spectent animalia cetera terram,
os homini sublime dedit, caelumque videre
iussit et erectos ad sidera tollere vultus.*
Ovid, Metamorphoses, I 84-86

1 Introduction

This old privilege notwithstanding, it took a long time to transform our gazing into the sky into physics. The study of the early universe truly left the mist of myth and speculation to become science only in the 1930's, with the discovery of cosmic expansion. We had to wait until the 1960's, with the discovery of the cosmic microwave background (CMB), to be able to discriminate between the Hot Big Bang and alternative cosmologies, and only in the early 1990's, with the detection of CMB's inhomogeneities, did cosmology fully become a quantitative science.

The celebrated WMAP survey [1] has spectacularly confirmed some general predictions of slow-roll inflation, and offered the possibility of significantly constraining alternative explanations for the primordial power spectrum. Future experiments may be able to go beyond the power spectrum and check for other features of the CMB, as, for instance, primordial non-Gaussianities.

Present data already make it meaningful to ask about finer details of the mechanism that generates an almost scale-invariant power spectrum. We just mentioned one such detail: possible non-Gaussian features. Another one is whether the correct initial state of the universe is the standard adiabatic “Bunch-Davies” vacuum [2]. Deviations from the standard inflationary vacuum offer the exciting possibility of observing very high-energy, “trans-Planckian” physics in the cosmic microwave background radiation, thanks to the enormous stretch in proper distance due to inflation [3]. This effect, which may present us with a real chance for probing string theory or any other model of quantum gravity, has received considerable attention, once the possibility was raised that these effects could be as large as H/M , with H the Hubble parameter during inflation, and M the scale of new physics (e.g. the string scale).

Due to our ignorance of the ultimate theory governing high-energy physics, the most natural, model-independent approach to studying modifications to the primordial power spectrum is effective field theory (EFT) [4, 5]. Using an EFT approach [4] concluded that the signature of any trans-Planckian modification of the standard inflationary power spectrum is $O(H^2/M^2)$, well beyond the reach of observation even in the most favorable scenario ($H \sim 10^{14}$ GeV, $M \sim 10^{16}$ GeV).

What was absent from e.g. ref. [4] was a systematic EFT approach to initial conditions. Ref. [4] presented convincing arguments against the (in)famous α -vacua [6] of de Sitter space, but it did not give a complete parametrization of finite-energy, non-thermal states. That parametrization was given in [7, 8], where the EFT approach was systematically extended to the choice of initial conditions. Refs. [7, 8] conclude that changes in the initial conditions for inflation are under control and may give $O(H/M)$ corrections to the primordial power spectrum. According to [7, 8], these corrections are quite characteristic of UV modifications to physics and

can be distinguished from other corrections arising instead from IR changes in the vacuum state.

In this paper, we shall argue that other constraints on the EFT of initial conditions make $O(H/M)$ changes in the primordial power spectrum unnatural, in the same sense that a light Fermi scale is unnatural in the Standard Model. Before explaining further this point, we need to sketch the approach of refs. [7, 8]. The most important difference between that and other approaches is that in [7, 8] initial conditions for modes of any wavelength are specified at the same initial time t^* . Other approaches give the initial conditions separately for each mode, at the time it crosses the horizon. The latter prescription is useful in the context of inflationary cosmology, but it obscures the field-theoretical meaning of the perturbation and/or initial condition: it does not easily account for the fact that after t^* curvatures and energy densities are small, so the field theory is under control, and it does not easily translate into an EFT language. The former prescription, instead, leads naturally to a simple classification of initial conditions in terms of *local* operators defined at the space-like boundary (i.e. initial surface) $t = t^*$. It also allows one to rephrase the question of naturalness of initial conditions for inflation in terms of the usual field-theoretical notion of naturalness.

In field theory we have a naturalness problem whenever the UV cutoff of the theory, M is much bigger than the observed value of the coefficients of unprotected relevant operators. For instance, a quadratically-divergent scalar mass term $m^2\phi^2$ is unnatural (or, equivalently, fine tuned) whenever $m \ll M$. In our EFT theory of initial conditions, we shall define a parallel notion. Initial conditions will be deemed unnatural whenever they require to fine-tune coefficients of relevant boundary operators to values much smaller than $M^{3-\Delta}$, where Δ is the dimension of the operator.

Section 2 is devoted to summarize the EFT approach of [7, 8]. There, we also generalize their formalism to the case where 3-d boundary interactions are not added at an “initial” time, but they are used instead to parametrize an unknown period in the history of the universe. We shall do that to apply EFT methods to ekpyrotic/cyclic [9] or pre-big bang cosmologies [10]. In a sentence: we shall replace the unknown physics at the bounce –where the scale factor shrinks to zero to originate a spacelike singularity– with a spacelike (instantaneous) brane, whose world-volume supports local operators and additional degrees of freedom (see fig. 1).

Section 3 explains in greater details the concept of naturalness for initial conditions. Naturalness is then applied to constrain possible changes in the B.D. vacuum of inflation. They turn out to be either unnatural or IR universal; therefore, ill-suited to characterize signals of new high-energy physics.

In section 4, we show that the spacelike brane that parametrizes the unknown Planckian physics at the bounce in cyclic/ekpyrotic/pre-big bang cosmologies can give raise to the correct power spectrum $P \sim k^{-3}$, generically at the price of severe non-localities on the brane. We

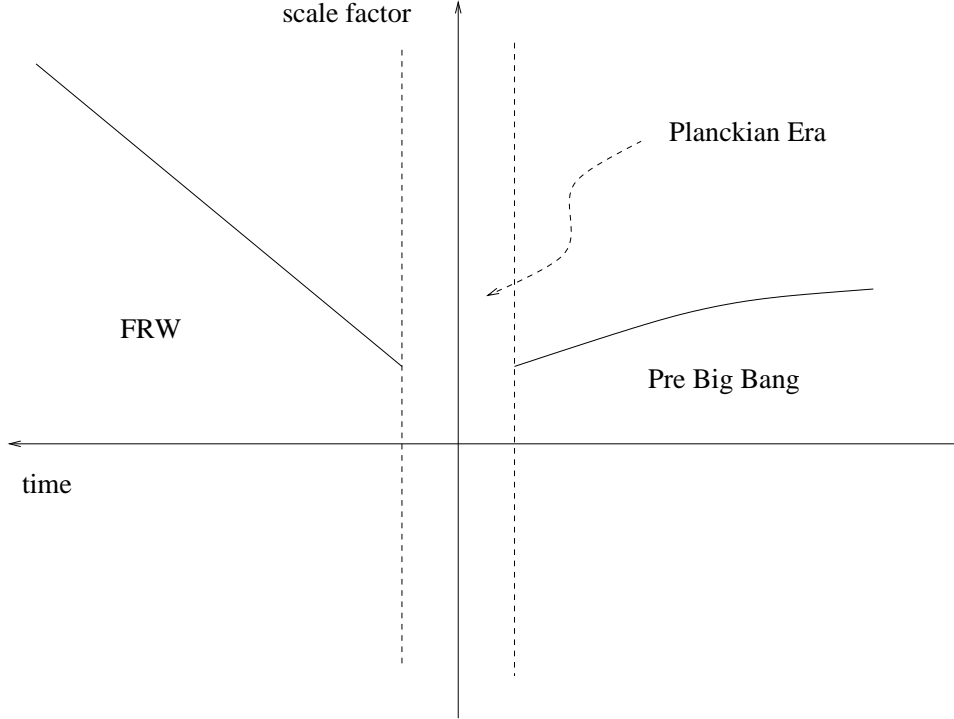


Figure 1: A schematic view of the EFT description of a cyclic cosmology. It plots the scale factor vs. time. The Planckian era around the bounce is replaced by an effective space-like brane (S-brane).

show that typically these non-localities lead after the bounce to faster-than light propagation of signals originating in the pre-big bang phase. This pathology may not be lethal for these models, since Lorentz invariance is explicitly broken by the brane. Nevertheless, it is troublesome. It is avoided with a particular choice of interaction terms on the brane, which we proceed to show is unnatural according to our general prescription, since it requires the fine-tuning to zero of an unprotected relevant operator. The rest of section 4 is used to show how to write in local form a non-local boundary interaction on the brane, which can generate the correct post-bounce power spectrum. We show that the boundary term can be written in terms of local operators, but only at the price of introducing extra auxiliary fields that propagate only on the spacelike brane. We then conclude the section by showing that, besides leading to faster-than-light propagation, a generic boundary term also requires to fine-tune certain relevant operators, and it is thus also unnatural. Our argument confirms and complements the analysis of [11] (see also [12]).

Section 5 shows how to compute the effect of changes in the initial state of inflation that do not affect the power spectrum, but that do change the three-point function of scalar fluctuations. We show that the changes induced by cubic boundary operators can lead to observable non-Gaussianities, different from those studied in [13], without requiring undue fine-tunings.

Section 6 studies the three point function of scalar fluctuations in cyclic cosmologies. Set-

ting aside naturalness considerations, we show that, in this scenario, any pre-big bang non-Gaussianity is damped. Finally, we show that the particular form of the scalar potential used in cyclic cosmologies sets a bound on how close to the singularity is the space-like brane. Specifically, if we want to avoid uncontrollably large interactions, the pre-big bang cosmic evolution must be cut off and replaced by an effective brane at a time parametrically larger than M_p^{-1} .

In section 7 we summarize our findings and conclude pointing out to possible developments of our formalism to other problems involving spacelike singularities.

2 EFT of Cosmological Initial Conditions

2.1 Inflation

Differently from other approaches, here as in [7, 8] we give initial conditions for modes of all wavelengths at the same initial time t^* .

The prescription starts by supplementing the EFT action describing all relevant low energy fields with a boundary term that encodes the standard thermal vacuum. To be concrete, we begin by working out the example of a massless scalar field in a time-dependent background. The 4d (bulk) action plus a 3d boundary term is

$$\begin{aligned} S &= S_4 + S_3, & S_4 &= \frac{1}{2} \int_{t^*}^{\infty} dt \int d^3x \sqrt{-g} g^{\mu\nu} \partial_\mu \chi^* \partial_\nu \chi, \\ S_3 &= \frac{1}{2} \int d^3x \sqrt{h^*(x)} \int d^3y \sqrt{h^*(y)} \chi^*(x) \kappa(x, y) \chi(y) \Big|_{t^*}. \end{aligned} \quad (1)$$

Here h_{ij}^* is the induced metric on the surface $t = t^*$. The role of S_3 is to specify the wave functional for the scalar χ at $t = t^*$:

$$\Psi[\chi(x)] = \exp(iS_3[\chi]). \quad (2)$$

Selecting an initial state for χ means in this language to choose a particular $\kappa(x, y)$. For instance, in de Sitter space with line element

$$ds^2 = a(\eta)^2 (-d\eta^2 + dx^i dx^i), \quad a(\eta) = -\frac{1}{H\eta}, \quad i = 1, 2, 3, \quad -\infty < \eta < 0, \quad (3)$$

the standard thermal [2] vacuum is obtained by choosing

$$\tilde{\kappa}(k) = -\frac{|k|^2 \eta^*}{1 + i|k|\eta^*}. \quad (4)$$

Here, a tilde denotes the Fourier transform from space coordinates to co-moving momenta k ($|k| \equiv \sqrt{k \cdot k}$) and η^* is the initial (conformal) time.¹ This expression for κ makes it clear that the choice of such initial time is conventional, since a change in η^* changes only κ , not the wave functional. From now on, whenever needed, the standard vacuum functional will be called $|0\rangle$.

¹From now on, η, η^* will denote the conformal time, t, t^* will denote the synchronous proper time, and $a()$ will always denote the scale factor. Notice that η^* here is negative.

2.2 Changing the Initial State

Next, we want to find a convenient classification of changes to the initial state. This can be done by adding a new boundary term to the action: $S \rightarrow S + \Delta S_3$. To determine ΔS_3 , we notice that, at any finite time t after t^* , we are insensitive to changes that only affect very low co-moving momenta k : co-moving momenta $|k| < H(t)a(t)$ correspond to perturbations with superhorizon physical wavelength $\lambda_p > 1/H(t)$, which are unobservable at time t . So, since we are interested in changes that can be observed in the CMB of the present epoch, we have an IR cutoff naturally built into the theory. This IR cutoff tells us that observable changes in the initial conditions can be parametrized by *local* operators:

$$\Delta S_3 = \sum_i \beta_i M^{3-\Delta_i} \int d^3x \sqrt{h^*} O^i \Big|_{t^*}. \quad (5)$$

Here O^i are operators of scaling dimension Δ_i , M is the high-energy cutoff of the EFT and the β_i 's are dimensionless parameters. The dimension Δ_i determines among other things how “blue” is the change in the power spectrum: the fractional change in the power spectrum is proportional to k^{Δ_i-2} . Since the EFT makes sense only for $k < M$, operators of high conformal dimension do not significantly change the observable spectrum. So, the most significant observable changes in the primordial fluctuation spectrum are parametrized by a few local operators of low conformal dimension.

We just mentioned that the EFT needs a UV cutoff. This means that the operators O_i must be suitably regulated at short distance. In other words, they are local only up to the cutoff scale M . As a simple example, consider the dimension-four operator $O^4 = (\beta/M)(\partial_i \chi)^2$. It has to be smeared at short distance, for instance by the replacement

$$\partial_i \chi \partial_i \chi \rightarrow \partial_i \chi f(-\partial^2/a^2(t^*)M^2) \partial_i \chi. \quad (6)$$

Here $f(x)$ is a smooth function obeying $f(x) = 1$, for $x \leq 1 - \epsilon$; $f(x) = 0$, for $x \geq 1 + \epsilon$; ϵ is a small positive number. The scale factor $a(t^*)$ appears because we want to cutoff at M the *physical* momentum $|k|/a(t^*)$, not the co-moving momentum $|k|$.

2.3 Power Spectrum

As a first application, let us derive the change in the power spectrum of a minimally-coupled scalar field in de Sitter space, induced by the operator O^4 introduced in the previous subsection [7]. The change in initial conditions ΔS_3 , is equivalent to perturbing the Hamiltonian of the system by an instantaneous interaction $H_I = -\delta(\eta - \eta^*)\Delta S_3$. So, the perturbed two-point correlation function is

$$G(k) = \lim_{\eta \rightarrow 0^-} \langle |\chi(k, \eta)|^2 \rangle = \lim_{\eta \rightarrow 0^-} \langle 0 | \exp(-i\Delta S_3) |\chi(k, \eta)|^2 \exp(i\Delta S_3) | 0 \rangle. \quad (7)$$

To first order in β , the change is

$$\delta G(k) = -i \frac{\beta}{M} \int d^3x a(\eta^*) \langle 0 | [O^4(x), |\chi(k, 0)|^2] | 0 \rangle. \quad (8)$$

This quantity is easily computed in terms of commutators of free fields in de Sitter space, resulting in [7, 14, 15]

$$\delta G(k) = -\frac{\beta}{M} a(\eta^*) \frac{H^2}{|k|^3} \text{Im} [\chi^+(\eta^*, k)]^2 |k|^2 f(|k|^2/a^2(\eta^*)M^2). \quad (9)$$

The canonically normalized, positive frequency solution of the free-field equations of motion is

$$\chi^+(\eta, k) = \frac{H}{\sqrt{2|k|^3}} (1 + i|k|\eta) \exp(-i|k|\eta), \quad (10)$$

and $\langle 0 | |\chi(k, 0)|^2 | 0 \rangle = H^2/|k|^3$ is the unperturbed two-point function. For $|k||\eta^*| \sim 1$, the effect of O^4 on the power spectrum, $P(k) = (k/2\pi)^3 G(k)$, can be as large as $\delta P/P \sim \beta H/M$, i.e. in the observable range when β is $O(1)$. For $|k||\eta^*| > 1$, the oscillating exponential gives a characteristic periodic signature [8]. The signal rapidly decays for $|k||\eta^*| \gg 1$, so, it can be detected only if inflation lasts for a relatively short period; otherwise, O^4 would only affect unobservable super-horizon fluctuation.

2.4 Scalar Metric Fluctuations

Of course, we are not really interested in the fluctuations of a minimally coupled scalar. Those describe at best tensor perturbations of the metric. We want instead scalar perturbations of the metric. In the standard setting for slow-roll inflation, where the field content is the metric $g_{\mu\nu}$ plus the inflaton ϕ . It is convenient to use the ADM formalism and decompose $g_{\mu\nu}$ into 3-d metric h_{ij} , shift N^i and lapse N :

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \quad (11)$$

Metric and scalar field split into background values plus fluctuations as $h_{ij} = \exp(2\rho)(\delta_{ij} + \gamma_{ij})$, $a(t) \equiv \exp(\rho)$, $\psi = \phi(t) + \varphi$. The last equation means that the background scalar field, $\phi(t)$, is a function of the time t only. A particularly convenient gauge is [13]

$$h_{ij} = \exp(2\rho + 2\zeta)(\delta_{ij} + \gamma_{ij}), \quad \partial_i \gamma_{ij} = 0, \quad \gamma_{ii} = 0, \quad \varphi = 0. \quad (12)$$

In this gauge, the scalar fluctuations are given by ζ , which is also the gauge invariant variable of [16] (see also [17]). The quadratic bulk action for ζ is almost identical with that of a massless scalar field

$$S = \frac{1}{16\pi G} \int dt d^3x \frac{\dot{\phi}^2}{\rho^2} [-e^{3\rho} \dot{\zeta}^2 + e^\rho (\partial\zeta)^2]. \quad (13)$$

$$= \frac{1}{16\pi G} \int d\eta d^3x \frac{\phi'^2}{\rho'^2} a^2(\eta) [-\zeta'^2 + (\partial\zeta)^2]. \quad (14)$$

For future reference, we have written the action both in synchronous time t and in conformal time η . Derivative w.r.t. t is denoted by a dot while a prime denotes derivation w.r.t. η ; G is the Newton constant.

This formula shows that scalar fluctuations can be treated as a non-canonically normalized scalar fields. So, the equations derived in the previous subsections translate into formulas for changes in the CMB. One may worry that this conclusion has been reached with a very special choice of gauge. This is not the case, though. In the 4-d bulk, in any gauge, only the combination $\zeta - (\dot{\phi}/\dot{\rho})\varphi$ propagates. This combination is indeed the gauge-invariant definition of the Bardeen variable (see [16] and [17] section 10.3), and it is this quantity that appears in the action. On the initial-time boundary, instead, both ζ and φ appear [17], but the latter appears only as a non-dynamical field, without a genuine kinetic term. Schematically:

$$S_{\text{boundary}} = \int d^3x F[\zeta(x, t^*), \varphi(x, t^*)], \quad (15)$$

where $F[\zeta, \varphi]$ is a quadratic function of its variables. Therefore, after imposing the non-dynamical φ equations of motion, $\partial F/\partial \varphi = 0$, φ becomes a linear function of ζ , that can be plugged back into the boundary term to yield a quadratic function of ζ only.

2.5 Cyclic Cosmologies

We shall try to obtain an EFT description of cyclic cosmologies, independent of the details of their unknown high-curvature phase. In our description, the period of cosmic evolution when perturbations are generated, separates into three periods (see fig. 1): a slow contraction, where the scale factor evolves as $a(\eta) = (-\eta/\eta^*)^\varepsilon$, $\eta < -\eta^*$, $\varepsilon \ll 1^2$; a high-curvature bounce at $|\eta| \approx \eta^*$, the unknown dynamics whereof is replaced by a surface term (a spacelike brane, or S-brane); a FRW phase, typically a radiation-dominated phase with $a(\eta) = \eta/\eta^*$, $\eta > \eta^*$. The scalar ϕ is no longer a slow-roll inflaton. In the pre-bounce, pre-big bang phase, the potential is a very steep exponential over some range of ϕ , for instance [18]:

$$V = -V_0 \exp(-\sqrt{16\pi G/p} \phi), \quad 0 < p \ll 1, \quad (16)$$

and the background is

$$a(\eta) = (-\eta/\eta^*)^{p/(1-p)}, \quad \phi(\eta) = \frac{\sqrt{2p/8\pi G}}{1-p} \log(-\eta/\eta^*), \quad (17)$$

where the parameters V_0 and η^* are related by $\eta^* = \sqrt{p(1-3p)/8\pi G V_0}$. After the bounce, the scalar decouples from the dynamics and one finds the action for scalar fluctuations in a radiation-dominated FRW cosmology.

²In the pre-big bang model of [10], instead, $\varepsilon = 1/2$. Here η^* is positive.

To match the two phases, we need to add a boundary term, an S-brane, with nonzero tension. Its form is largely free; we can only say that it must contain at least the following terms:

$$T_+ \int d^3x \sqrt{h_+^*} - T_- \int d^3x \sqrt{h_-^*} + \dots \quad (18)$$

Here T_+ and T_- are constant tensions, needed to satisfy the junction conditions at $\eta = \eta^*$ and $\eta = -\eta^*$, respectively. h_\pm^* is the determinant of the induced metric on the surface $\eta = \pm\eta^*$ and ... stands for gradient terms that vanish on the background $\zeta = 0$.

The nonzero tensions generate a stress-energy tensor, which is conserved but which violates the null energy condition (NEC). This is unavoidable if we want a spatially flat bounce.

Thanks to the S-brane, we can have a bounce and we can write the action for scalar fluctuations in the gauge (12) as

$$\begin{aligned} 16\pi GS &= \int_{\eta^*}^{\infty} d\eta \int d^3x \left(\frac{\eta}{\eta^*} \right)^2 [-\zeta'^2 + (\partial\zeta)^2] + \frac{2}{p} \int_{-\infty}^{-\eta^*} d\eta \int d^3x \left(\frac{-\eta}{\eta^*} \right)^{2p/(1-p)} [-\zeta'^2 + (\partial\zeta)^2] \\ &+ \int d^3x \{ \Lambda[\zeta(\eta^*) - \alpha(\partial)\zeta(-\eta^*)] + \zeta(\eta^*)F(\partial)\zeta(\eta^*) \}. \end{aligned} \quad (19)$$

This is one of the most important equations in our paper, and it is worth of a few comments.

First of all, the boundary term parametrizing the unknown high-energy physics at the bounce is made of two pieces. The first is the one containing the Lagrange multiplier $\Lambda(x)$. This is a non-dynamical field whose role is to link the value of the field ζ after the bounce, at $\eta = \eta^*$, to its value before the bounce, at $\eta = -\eta^*$. ζ can be continuous at the bounce, for $\alpha = 1$, or it can jump, whenever $\alpha \neq 1$. $\alpha(\partial)$ is a local function of spatial gradients. So, $\zeta(\eta^*)$ is generically a function of $\zeta(-\eta^*)$ and its derivatives along the brane. Likewise, $F(\partial)$ is a function of spatial gradients. Generically, it is nonlocal. Its role is to mimic super-horizon correlations induced by the unknown physics at the bounce. Superhorizon correlations do not necessarily signal acausalities, because our parametrization can also fit a “long” bounce, lasting for an arbitrarily long time.

Notice that we did not introduce $d\zeta/d\eta$ in the boundary term. The reason is that $d\zeta/d\eta$ terms render the variation of eq. (19) ill-defined. So, whenever they do appear in a boundary term, they should be eliminated by an appropriate discontinuous field redefinition [7], or by using the bulk equations of motion to convert them into functions of ζ and its spacelike gradients.

We wrote eq. (19) in a specific gauge. To write it in a 3-d covariant form, we must re-express ζ in terms of the intrinsic curvature on surfaces defined by $\varphi = 0$. To write it in a fully 4-d covariant manner, we must furthermore express the position of the bounce in terms of a covariant (scalar) equation involving, say, ψ and the scalar curvature.

We may worry about two features of this procedure.

1. A covariant equation for the position of the bounce reduces, in the gauge (12), to

$$\eta = G(\zeta) + \eta^*. \quad (20)$$

The function G is smooth and it vanishes at $\zeta = 0$, but is otherwise unknown. So, generically, the bounce does not sit at a constant value of the time $\eta = \eta^*$. This is not a problem since at quadratic order this bending of the brane only generates further quadratic terms of the form $\zeta F \zeta$, plus a finite renormalization of the brane tensions T_{\pm} .³ Explicitly, the induced metric on the brane is

$$h_{ij}^* = \partial_i G \partial_j G g_{00}(G(x), x) + \partial_i G g_{0j}(G(x), x) + \partial_j G g_{0i}(G(x), x) + g_{ij}(G(x), x). \quad (21)$$

By substituting this formula into the universal term $T \int d^3x \sqrt{h^*}$ we get, at linear order in ζ

$$T \int d^3x a^3 (3\zeta + 3\rho' G_{\zeta} \zeta), \quad G_{\zeta} \equiv \left. \frac{dG}{d\zeta} \right|_{\eta=G(x)}. \quad (22)$$

This term induces a finite renormalization of the brane tension, $T \rightarrow T(1 + \rho' G_{\zeta})$. At quadratic order in ζ we find as announced a term of the form $\zeta F \zeta$, after eliminating time derivatives by the bulk equations of motion:

$$T \int d^3x a^3 \left[G_{\zeta}^2 (\partial_i \zeta)^2 + 2G_{\zeta} \partial_i \zeta N_i a^{-2} + 3(2\rho'^2 + \rho'') G_{\zeta}^2 \zeta^2 + 6\zeta^2 + 6G_{\zeta} \zeta \zeta' + 3\rho' G_{\zeta \zeta} \zeta^2 + \frac{3}{4} (2\zeta + 2\rho' G_{\zeta} \zeta)^2 \right]. \quad (23)$$

2. The functions α and F we introduced, are quite arbitrary. They can even break explicitly 3-d rotations and translations, even though we will not do that in the following. So, neither the momentum constraint nor the Hamiltonian constraint impose any restriction on them. This is because our “initial time” brane is different from the boundary brane introduced in the context of de Sitter holography in [19]. For us the brane is real. It either parametrizes a very specific time in cosmic evolution (the bounce), or a specific choice of quantum state at some time early in the inflationary epoch. Since we use the brane to specify an initial state, which is then used to compute expectation values of ζ at $\eta \rightarrow \infty$ (that is today), we can change the value of η^* , together with the form of the boundary terms, in such a way as to keep the late-time correlators constant. An example of this “renormalization group” is our eq. (4). So, our boundary terms at η^* are the analog of *arbitrary* initial conditions for the RG group. In ref. [19] instead, the late-time brane is a regulator, and the time evolution of all fields is fixed, because both late-time and early-time boundary conditions are fixed. The difference between the two approaches is schematically shown in fig. 2.

³The brane bending also introduces terms in ζ' . As we have already mentioned, these terms must be canceled by field redefinitions, or expressed in terms of gradients via the equations of motion.

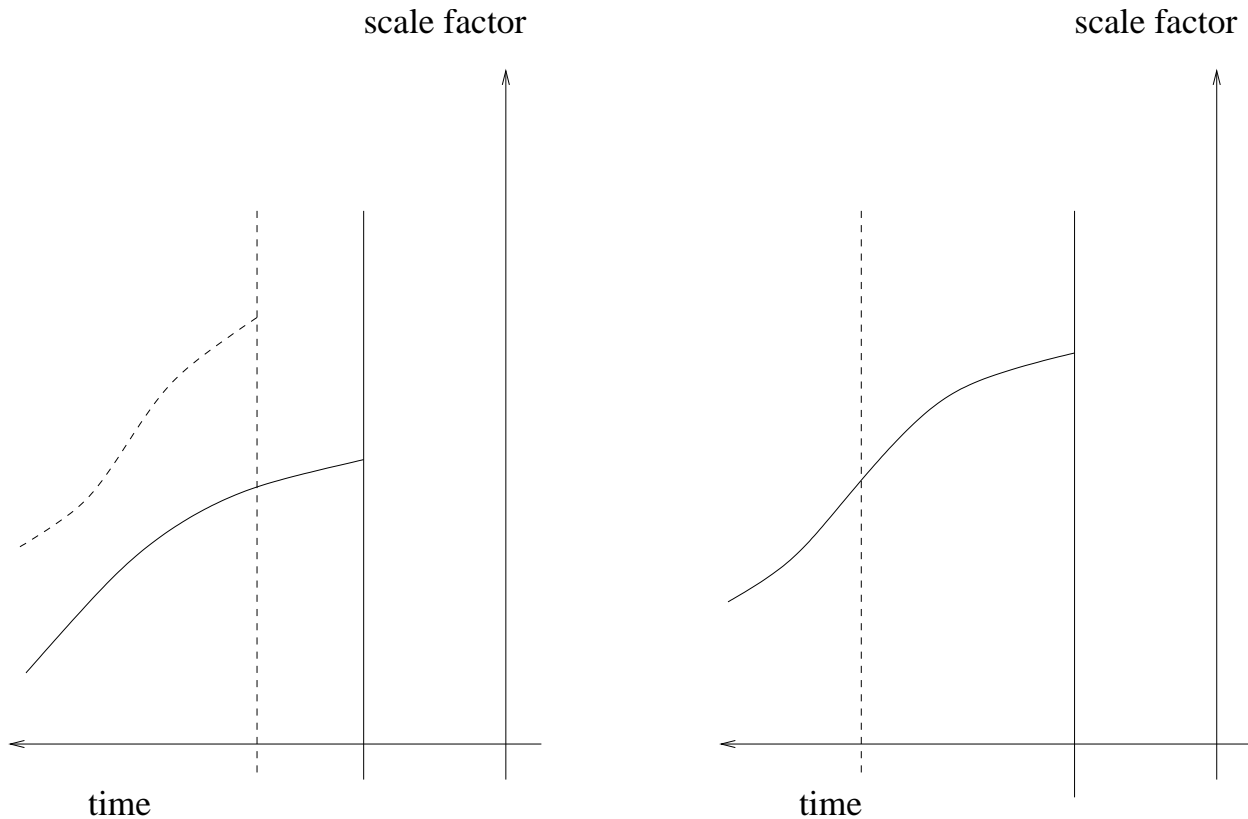


Figure 2: On the left, a schematic view of our brane, its position is represented by the vertical lines. By changing the operators on the brane and/or its position, we can change the initial conditions for the flow, hence the flow itself. On the right, the holographic brane of ref. [19]. Its boundary operators are constrained by requiring that the flow remains the same when its position is changed.

3 Naturalness

In particle physics, stability under radiative corrections is a powerful guide for constraining high-energy extensions of the standard model or of any EFT. By construction, an EFT has a built-in UV cutoff. In other words, the EFT only describes the low-energy sector of a theory whose UV completion is unknown. EFT is a powerful method whenever there is a large energy gap between a known low-energy sector (say the standard model) and an unknown high-energy sector –typically made of heavy particles– that decouples below a cutoff M . In this case, the effect of integrating out the high-energy sector is to introduce irrelevant operators in the EFT. They have dimension $\Delta > 4$ and appear in the EFT with coefficients $O(M^{4-\Delta})$. A change in the UV physics –as for instance a change in masses and couplings of the heavy sector– induces a small modification in the coefficients of these irrelevant operators. So, low-energy physics is shielded from changes in the UV physics, and EFT can be predictive even in the absence of a complete knowledge of high-energy physics. The problem lies in the relevant operators; those

with dimension $\Delta < 4$. Their coefficients must be much smaller than $M^{4-\Delta}$, otherwise, the EFT would describe a trivial physics with no light states at all. A simple example of this pathology is a theory with two scalars, one heavy, with mass M , and another one with mass m . Below the energy scale M only one scalar propagates, and only if $m \ll M$. Relevant operators are very sensitive to changes in the UV physics, the more so the lower their dimension. Generically, they appear in the EFT with coefficients $O(M^{4-\Delta})$. Even if the tree-level values of these parameters are small, radiative corrections typically bring them to values $O(M^{4-\Delta})$. Equivalently, any small change in masses, couplings etc. in the heavy sector bring these coefficients to their typical value.

The best known naturalness problem in particle physics is that associated with the Fermi scale $M_F \approx 100 \text{ GeV}$ or, equivalently, the Higgs mass. Suppose that the standard model holds up to a high energy scale $M \approx 10^{16} - 10^{19} \text{ GeV}$. Then, unless the Higgs mass is protected by symmetries (e.g. supersymmetry) it would be destabilized by radiative corrections, and driven to the UV cutoff.

In our setting, we have a naturalness problem too. It arises because we change the cosmological initial state by adding boundary terms to a bulk action as in eq. (5). These boundary terms are generically unstable under radiative corrections. So, even if we begin by modifying our theory by adding a (safe) irrelevant boundary operator⁴, we may end up generating dangerous relevant operators.

3.1 Naturalness in Inflation I

We begin by studying in detail the example given in section 2.3. There we computed the change induced in the fluctuation spectrum of a minimally-coupled massless scalar by the dimension-4 operator

$$O^4 = \frac{\beta}{M} (\partial_i \chi)^2. \quad (24)$$

This boundary term is not complete. It must be covariantized with respect to 3-d general coordinate transformations, which are not broken by a choice of initial conditions. The covariantization is obvious and gives the following boundary term

$$\Delta S_3 = \frac{\beta}{M} \int d^3x \sqrt{h} h^{ij} \partial_i \chi \partial_j \chi. \quad (25)$$

Once we impose a shift symmetry $\chi \rightarrow \chi + \text{constant}$, O^4 is the lowest-dimension boundary operator quadratic in χ . So, it is not unnatural to set to zero the coefficient of the relevant –and potentially dangerous– operator $O^2 = \alpha M \chi^2$.

⁴Since the boundary is a 3-d field theory, all operators of dimension greater than 3 are irrelevant.

In the gauge (12), the boundary term (25) gives rise to interaction terms between χ and ζ . In particular, we get a quartic interaction

$$\Delta S_3^{\zeta^2\chi^2} = \frac{\beta}{2M} \int d^3x a(\eta^*) \zeta^2 \partial_i \chi \partial_i \chi. \quad (26)$$

This is a dangerous interaction. Its effect is best seen by writing the wave function at late time as a functional integral

$$\Psi[\zeta_{now}, \chi_{now}] = \int [d\zeta d\chi] e^{i(S_4 + S_3 + \Delta S_3)}. \quad (27)$$

χ, ζ are given free boundary conditions at the initial time η^* , while at late time, $\eta \rightarrow 0^-$, they are given fixed (Dirichlet) boundary conditions

$$\zeta(\eta)|_{\eta=0} = \zeta_{now}, \quad \chi(\eta)|_{\eta=0} = \chi_{now}. \quad (28)$$

This functional integral can be computed perturbatively either in terms of Feynman diagrams or using the Hamiltonian formalism of subsection 2.3. Results become clearer using the Feynman diagram approach.

First of all, bulk interactions may be potentially dangerous, since naively the bulk action has interaction terms of the form $(16\pi G)^{-1} \zeta \partial \zeta \partial \zeta$, while the kinetic term is multiplied by the extra factor $\epsilon = \phi'^2/\rho'^2$ [see eq. (14)], which is small in slow-roll inflation. Nevertheless, as shown in [13], a field redefinition of ζ eliminates all such terms and leaves only bulk terms of the form $(16\pi G)^{-1} \epsilon^2 \zeta \partial \zeta \partial \zeta$.

The boundary interaction (26), instead, does produce a dangerous effect. By integrating out χ , this interaction generates a new boundary term for ζ thanks to the diagram in fig. 3.

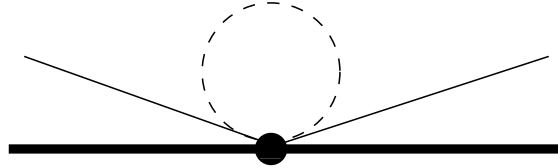


Figure 3: Self-energy correction to the ζ^2 boundary “mass” terms. ζ ’s are denoted by solid lines and χ ’s by broken lines.

$$\Delta S^{\zeta^2} = \frac{\beta}{2M} \int d^3x a(\eta^*) \zeta^2 \langle \partial_i \chi \partial_i \chi \rangle. \quad (29)$$

The propagator for χ is the standard one, computed in the BD vacuum

$$\langle \chi(\eta, k) \chi(\eta', -k) \rangle = \frac{H^2}{2|k|^3} (1 + i|k|\eta)(1 - i|k|\eta') \exp[i|k|(\eta' - \eta)], \quad \eta \geq \eta'. \quad (30)$$

By substituting this expression, computed at $\eta = \eta^*$, into eq. (29) we get

$$\Delta S^{\zeta^2} \approx \frac{\beta}{2M} \int d^3x a(\eta^*) \zeta^2 \int_{\frac{|k|}{a(\eta^*)} \leq M} \frac{d^3k}{(2\pi)^3} \frac{H^2}{2|k|^3} |k|^2 (1 + |k|^2 \eta^{*2}) \approx \frac{\beta}{96\pi^2} M^3 \int d^3x a^3(\eta^*) \zeta^2. \quad (31)$$

Here we used a sharp momentum cutoff and the inequality $M \gg H$. Any other cutoff would give a like result.

The effect of this term on the power spectrum is enormous, since the kinetic term for scalar fluctuations is multiplied by the small number ϵ . If we canonically normalize ζ by

$$\zeta = \sqrt{\frac{8\pi G}{\epsilon}} v, \quad (32)$$

we can use the same technique as in subsection 2.3 to get

$$\frac{\delta P(k)}{P(k)} = C a^3(\eta^*) \text{Im}[v^+(\eta^*, k)]^2, \quad v^+(\eta, k) = \frac{H}{\sqrt{2|k|^3}} (1 + i|k|\eta) \exp(-i|k|\eta), \quad \frac{|k|}{a(\eta^*)} \ll M. \quad (33)$$

With our normalization, $C = \beta G M^3 / 12\pi\epsilon$. So, for $k \sim 1/\eta^*$, the power spectrum receives corrections $O(\beta G M^3 / 12\pi H \epsilon)$. By imposing the observational constraint $\delta P(k)/P(k) \lesssim \epsilon$ we arrive at our first naturalness constraint on β

$$\beta \lesssim 12\pi\epsilon^2 (G M^2)^{-1} \frac{H}{M}. \quad (34)$$

Notice that the power corrections we computed in section 2.3 were at most $O(\beta H/M)$. Since the term (31) is also linear in β and as large as $\beta G M^3 / 12\pi H \epsilon$, it dominates over the tree-level term whenever $G M^4 / 12\pi\epsilon H \gtrsim 1$. Constraints of this kind were derived with a different method in [14, 15]. The method used here makes it clear that the bound comes from asking that the correction to the BD initial state is generic and robust under radiative corrections. Of course, as with any divergence, even that in eq. (31) can be canceled by appropriately choosing boundary counter-terms. The point is that this choice implies a fine-tuned UV completion of our EFT, which we should not assume without a valid reason such as a symmetry, or a better knowledge of the UV physics.

Notice that the counter-term that cancels (31) is not the same that renormalizes the brane tension. The brane tension is renormalized because when we expand eq. (25) in powers of ζ , we also get a linear term

$$\Delta S_3^{\zeta\chi^2} = \frac{\beta}{M} \int d^3x a(\eta^*) \zeta \partial_i \chi \partial_i \chi. \quad (35)$$

This term can be canceled by changing the tension of the term

$$T \int d^3x \sqrt{h} = T \int d^3x a^3(\eta^*) e^{3\zeta} = T \int d^3x a^3(\eta^*) \left(1 + 3\zeta + \frac{9}{2}\zeta^2 + \dots\right). \quad (36)$$

Equation (31) implies we can cancel the linear term (35) by changing the tension as

$$3\delta T + \frac{\beta}{48\pi^2} M^3 = 0. \quad (37)$$

To cancel the quadratic term, instead, we would need to change the tension as

$$9\delta T + \frac{\beta}{48\pi^2} M^3 = 0. \quad (38)$$

3.2 Naturalness in Inflation II

In the previous subsection, we introduced an extra scalar field, χ , besides the inflaton. This was done to simplify our analysis and is by no means necessary. We could have introduced a boundary interaction involving only ζ , for instance⁵

$$\Delta S_3 = \frac{\gamma}{M} \frac{\epsilon}{8\pi G} \int d^3 x a(\eta^*) \zeta \partial^2 \zeta. \quad (39)$$

This boundary term can be covariantized by recalling that at linear order the 3-d Ricci curvature \mathcal{R}_{ij} is proportional to $\partial_i \partial_j \zeta$ plus a term linear in the transverse traceless fluctuation γ_{ij} . So, the 3-d covariant form of eq. (39) becomes

$$\Delta S_3 = \frac{\gamma}{M} \frac{\epsilon}{8\pi G} \int d^3 x \sqrt{h} \mathcal{R}. \quad (40)$$

Besides the quadratic term (39), eq. (40) produces, among others, a quartic interaction term. In terms of the canonically normalized field v it reads

$$\Delta S_3^{v^4} = \frac{4\pi G \gamma}{M \epsilon} \int d^3 x \frac{1}{a(\eta^*)} v^3 \partial^2 v. \quad (41)$$

The self-energy loop now produces a boundary term of the form

$$\frac{\gamma G M^3}{9\pi \epsilon} \int d^3 x a^3(\eta^*) v^2. \quad (42)$$

Following the same steps we used to arrive to eq. (34) we obtain a constraint on γ

$$\gamma \lesssim 9\pi \epsilon^2 (GM^2)^{-1} \frac{H}{M}. \quad (43)$$

Covariantization of the boundary terms is just one of many ways in which dangerous boundary interactions appear. In the next subsection we will show that such interactions are induced at the one-loop level when we include the cubic vertices of the bulk theory. One interesting

⁵The factor $\epsilon(8\pi G)^{-1}$ is included for convenience so that for the canonically normalized field v the coefficient reduces to γ/M .

source of boundary interactions is the field redefinition needed to make all bulk self-interactions of ζ $O(\epsilon^2)$. As it was pointed out in [13], all $O(\epsilon)$ bulk cubic interactions assemble into the form

$$S_4^{\zeta^3} = \frac{\epsilon}{8\pi G} \int d\eta d^3x f(\zeta) \left[-\frac{d}{d\eta} a^2(\eta) \frac{d}{d\eta} \zeta + a^2(\eta) \partial^2 \zeta \right], \quad (44)$$

where $f(\zeta)$ is a quadratic function of ζ . Since they vanish on the linearized equations of motion, in brackets, they can be canceled by a local field redefinition of the form $\zeta \rightarrow \zeta + f(\zeta)$ [13]. When substituted in eq. (39), this field redefinition generates quartic boundary interactions

$$\Delta S_3^{\zeta^4} = \frac{\gamma}{M} \frac{\epsilon}{8\pi G} \int d^3x a(\eta^*) f(\zeta) \partial^2 f(\zeta). \quad (45)$$

The function $f(\zeta)$ is given in [13], and we shall use its explicit form in the next subsection, to estimate the size of induced boundary interactions. Here, it suffices to notice that it contains, among others, terms like $(a/2a')^2 (\partial\zeta)^2$, which contain no factors of ϵ . So, the interaction (45) can generate a large boundary mass term through a diagram as in figure 4.

To sum up, small corrections by seemingly benign irrelevant boundary operators generically induce by radiative corrections large, dangerous, relevant boundary operators. This problem does not signal an outright inconsistency of non-BD initial conditions for inflation, but it makes their description in terms of an EFT unnatural, that is very sensitive to its UV completion.

An equivalent way of stating the problem is simply that the first correction to be expected in a generic EFT of initial conditions is

$$\alpha M \int d^3x a^3(\eta^*) v^2, \quad (46)$$

where α is a small coefficient at most of order $\epsilon H/M$ (because of experimental constraints!). This is the first universal correction to the BD vacuum that generically dominates over all others. Any UV modification of physics, be it strings, non-Lorentz invariant dispersion relations, or simply phase transitions in the late stages of inflation, reduces to this same term. In this perspective, modifications of the primordial power spectrum seem ill suited to discriminate among different types of new high-energy physics.

We could have guessed this result because (46) is the lowest-dimension *local* operator that can be written with the field v and its derivatives. The only question is whether this operator is generated after all. Our explicit calculation answered in the affirmative. There is one last subtlety to explain. Locality depends on the variable we choose to parametrize scalar fluctuations. The correct one is the canonical scalar field v , which has scaling dimension one. It is in terms of this field that one finds that radiative corrections renormalize the coefficients of relevant operators as $M^{3-\Delta}$ etc. We can rewrite (46) or any other local operator in v in manifestly covariant form using the 3-d metric h_{ij} . This expression need not be local in h_{ij} . One possible

covariantization of (46) is

$$\alpha M \frac{\epsilon}{8\pi G} \int d^3x \sqrt{h} \mathcal{R} \Delta^{-2} \mathcal{R}, \quad (47)$$

where Δ is the covariant scalar Laplacian in 3-d.

3.3 Naturalness in Inflation III

The strongest naturalness constraints on the coefficients of the boundary irrelevant operator in (eq. 39) arise when we take into account the boundary nonlinear self-interactions of the field ζ . In this section we show that the constraints obtained in this way are the same as those obtained in ref. [15] from backreaction considerations.

In [13], the cubic interaction terms of the field ζ were found to be

$$S^{\zeta^3} = \frac{\epsilon}{8\pi G} \int d^3x d\eta a^2(\eta) \left[\epsilon \zeta (\zeta')^2 + f(\zeta) \square \zeta + \dots \right]. \quad (48)$$

Here $\square \zeta$ is the term inside brackets in eq. (44),

$$f(\zeta) = \epsilon \zeta^2 + \frac{a^{-1}(\eta)}{H} \zeta \zeta' + \frac{a^{-2}(\eta)}{H^2} (\partial \zeta)^2, \quad (49)$$

and we have omitted terms of higher order in ϵ . These cubic interactions give UV divergent Feynman diagrams that induce new marginal *and* relevant boundary operators of the form

$$O^0 = \tilde{\gamma} \frac{\epsilon}{8\pi G} \int d^3x a^2(\eta^*) \zeta \zeta', \quad O^1 = \alpha M \frac{\epsilon}{8\pi G} \int d^3x a^3(\eta^*) \zeta^2. \quad (50)$$

The operator $O^{(0)}$ must be interpreted as explained in section 2.5: the $\zeta \zeta'$ term on the boundary is incompatible with a consistent variational principle, so it has to be eliminated with a field redefinition. Equivalently, if we work only to linear order in $\tilde{\gamma}$, it can be transformed into a term of the type ζ^2 by using the unperturbed BD boundary condition, $\zeta'(\eta^*) = \kappa \zeta(\eta^*)$.

The same procedure used in section 2.3 shows that these boundary operators lead to modifications of the primordial power spectrum of the form

$$\frac{\delta P^{(0)}(k)}{P(k)} \sim \tilde{\gamma} g^{(0)}(k\eta^*), \quad \text{and} \quad \frac{\delta P^{(1)}(k)}{P(k)} \sim \alpha \frac{M}{H} g^{(1)}(k\eta^*), \quad (51)$$

where $g^{(0)}$ and $g^{(1)}$ are both $O(1)$.⁶

The clearest way to exhibit the renormalization effect of the cubic interaction on the boundary Lagrangian is to perform a quadratic field redefinition on ζ , of the form $\zeta \rightarrow \zeta + f(\zeta)$ [13]. This redefinition eliminates from the bulk Lagrangian the interaction proportional to $f(\zeta)$, leaving only cubic terms of $O(\epsilon^2)$ in the bulk. On the boundary, on the other hand, the effect of the field redefinition is to produce new cubic and quartic interactions:

$$\int d^3x a(\eta^*) \zeta \partial^2 \zeta \rightarrow \int d^3x a(\eta^*) [\zeta + f(\zeta)] \partial^2 [\zeta + f(\zeta)]. \quad (52)$$

⁶Explicitly: $g^{(0)}(y) = \cos y - y^{-1} \sin y$, $g^{(1)}(y) = y^{-2} [2 \cos y - (1 - y^2) y^{-1} \sin 2y]$.

Using the explicit form of $f(\zeta)$, eq. (49), we see that new boundary action contains, among others, the terms

$$\Delta S_1^b = \frac{\gamma}{M} \frac{\epsilon}{8\pi G} \int d^3x \frac{\epsilon}{H} \zeta^2 \partial^2 (\zeta \zeta'), \quad \Delta S_2^b = \frac{\gamma}{M} \frac{\epsilon}{8\pi G} \int d^3x \frac{1}{a^2(\eta^*) H^3} \zeta \zeta' \partial^2 (\partial \zeta)^2. \quad (53)$$

These interactions induce boundary operators of the form (50), through the Feynman diagram shown in fig. 4.

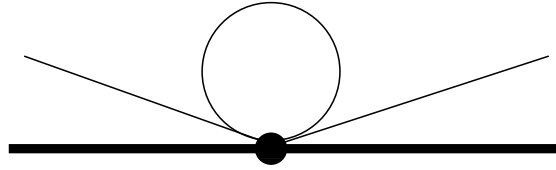


Figure 4: Effective boundary two-point vertex induced at one loop by the field redefinition $\zeta \rightarrow \zeta + f(\zeta)$. The quartic interaction is proportional to $\gamma f \partial^2 f$.

By plugging the interaction ΔS_1^b in the vertex of this diagram we arrive at a contribution to the effective action for ζ :

$$\Delta S^{\zeta^2} \approx \frac{\epsilon}{8\pi G} \frac{\gamma}{M} \frac{\epsilon}{H} \int \frac{d^3q}{(2\pi)^3} \zeta^2(q, \eta^*) \int \frac{d^3k}{(2\pi)^3} k^2 \langle \zeta(k, \eta^*) \zeta'(-k, \eta^*) \rangle \quad (54)$$

$$\approx \epsilon H \frac{\gamma}{M} \int \frac{d^3q}{(2\pi)^3} \zeta^2(q, \eta^*) \int_{|k| \leq M a(\eta^*)} \frac{d^3k}{(2\pi)^3} k^2 \eta^{*2} \quad (55)$$

$$\approx \epsilon \gamma \frac{M^4}{H} \int d^3x a^3(\eta^*) \zeta^2(\eta^*), \quad (56)$$

where in the third line we have considered only the most divergent term, and we used for ζ the scalar field propagator given in eq. (30), up to an normalization factor $8\pi G \epsilon^{-1}$:

$$\langle \zeta(k, \eta) \zeta(-k, \eta') \rangle = \frac{8\pi G}{\epsilon} \langle \chi(k, \eta) \chi(-k, \eta') \rangle. \quad (57)$$

The induced boundary term is the operator $O^{(1)}$ in eq. (50), with a coefficient

$$\alpha \sim \gamma \frac{8\pi G M^3}{H}. \quad (58)$$

A similar calculation shows that, by plugging the interaction ΔS_2^b in (53) in the vertex of the diagram in fig. 4, one generates the operator $O^{(0)}$ given in eq. (50), with a coefficient

$$\tilde{\gamma} \sim \frac{\gamma}{\epsilon} \frac{8\pi G M^5}{H^3}. \quad (59)$$

By asking that $\delta P^{(1)}(k)/P(k)$ and $\delta P^{(0)}(k)/P(k)$ are within the presently acceptable deviation from scale invariance in the power spectrum, i.e. that they are smaller than the combination of

slow-roll parameters ($\epsilon + \eta$), we get, from eq. (51), the constraints found in [15] for the coefficient γ :

$$\gamma \lesssim \epsilon(GM^2)^{-1} \frac{H^2}{M^2}, \quad \gamma \lesssim \epsilon^2(GM^2)^{-1} \frac{H^3}{M^3}, \quad \gamma \lesssim \epsilon\eta(GM^2)^{-1} \frac{H^3}{M^3}. \quad (60)$$

Again, we want to stress that these are naturalness bounds, in that they can be avoided by tuning the coefficient of the boundary operators in eq. (50) to a much smaller value than they would generically have in the presence of the boundary perturbation, eq. (39).

3.4 Coda: Is the BD Vacuum Natural?

In the previous subsections we applied standard techniques borrowed from field theory to study the naturalness of initial conditions that, below the cutoff energy M , differ from those specified by the Bunch-Davis vacuum. One might wonder what happens in the absence of these modifications: in particular, one might worry that the methods we used make even the unperturbed BD vacuum fine tuned, thus making the naturalness bounds we found less interesting. This would happen if one could find Feynman diagrams producing large boundary renormalizations that do not arise from the perturbation γ , i.e. they do not vanish at $\gamma = 0$. This is not the case, if one uses as the unperturbed vacuum, i.e. the one corresponding to $\gamma = 0$ in eq. (39), the vacuum of the *interacting* theory. Indeed, the interacting BD vacuum is defined by the functional integral

$$\Psi_{BD}[\zeta] = \int [d\zeta \dots] \exp(iS), \quad (61)$$

where $S = \int d\eta L$ is the action of the *interacting* theory, and the integration in η runs from η^* to $\eta = -\infty + i\epsilon$. The Euclidean continuation in η selects the BD (or Hartle-Hawking [20]) wave function. In the language of ref. [7] we are thus choosing “transparent” boundary conditions. Now, the wave function at any later time $\eta' > \eta^*$ is also defined by the functional integral (61), with the range of integration ranging from η' to $-\infty + i\epsilon$. In this representation, there is no boundary at η^* , so the field redefinition will not produce any large boundary term.

By splitting the integration in two regions, before and after the “initial” time η^* , this obvious result is reinterpereted as a cancelation between boundary terms arising from the field redefinition of the bulk action $\int_{\eta > \eta^*} d\eta L$, and the boundary terms defining the BD vacuum. So, the only nonlinear interactions left will be bulk terms suppressed by powers of the slow-roll parameter. This result shows that the slow-roll inflation background is stable against both bulk and boundary radiative corrections. In the presence of the boundary perturbation (39), however, this argument breaks down, and new boundary terms containing $f(\zeta)$ are generated by the field redefinition.

Finally, we should point out again another important feature that distinguishes the BD vacuum from all others. As we have seen, a modification of the BD vacuum by new physics at

a physical energy scale M generically manifests itself in the EFT by the boundary term (46). Generically, one would expect the coefficient α to be $O(1)$, yet observation constrains it to be smaller than $O(\epsilon H/M)$. Since this constraint comes from the size of $\delta P(k)/P(k)$ at $|k| = |\eta^*|^{-1}$, i.e. at $|k|_{phys} = |k|/a(\eta^*) = H$, an alternative route to make the correction (46) compatible with experiment is to make inflation last long enough to stretch $|k|_{phys}$ above today's horizon scale. This requires $a(\eta_{now})/a(\eta^*) > H/H_{now}$. In other words, the change in the initial state of inflation behaves exactly as any other pre-inflation inhomogeneity. Moreover, its effect is generically described almost entirely by a single operator, eq. (46), no matter what originated the change. So, the BD vacuum is natural also thanks to the very property that defines inflation: wait long enough, and all changes will be diluted away.

4 Naturalness in Cyclic Cosmologies

4.1 Scale-Invariant Spectrum

Ref. [11] convincingly showed that a scale-invariant spectrum of fluctuations cannot be produced before the bounce in the contracting phase described by eq. (17) with $p \ll 1$. So, it must be created at the bounce. Both effects can be seen very clearly in our EFT formalism.

We parametrized the unknown Planckian physics at the bounce in a cyclic cosmology in eq. (19). By requiring that eq. (19) is stationary under arbitrary variations in ζ and Λ , including those that do not vanish at the bounce, we get the equations of motion for ζ as well as the junction conditions at the bounce

$$\zeta(\eta, x) \equiv \int \frac{d^3k}{(2\pi)^3} \zeta(\eta, k) e^{ikx}, \quad \zeta'' + 2\frac{b'}{b}\zeta' + k^2\zeta = 0, \quad (62)$$

$$b(\eta) = \frac{\eta}{\eta^*} \quad \text{for } \eta \geq \eta^*, \quad b(\eta) = \sqrt{\frac{2}{p}} \left(\frac{-\eta}{\eta^*} \right)^{p/(1-p)} \quad \text{for } \eta \leq \eta^*,$$

$$\zeta(\eta^*, k) = \alpha(k)\zeta(-\eta^*, k), \quad \left(\frac{d}{d\eta}\zeta + p\alpha(k)\Lambda \right) \Big|_{-\eta^*} = 0, \quad (63)$$

$$\left(\frac{d}{d\eta}\zeta + F(k)\zeta + \Lambda \right) \Big|_{\eta^*} = 0. \quad (64)$$

Before the bounce, the metric can be well approximated by a Minkowski metric, for $p \ll 1$, and the general solution to the ζ equations of motion is approximately a plane wave

$$\zeta(\eta, k) = Ae^{-i|k|(\eta+\eta^*)} + Be^{+i|k|(\eta+\eta^*)}. \quad (65)$$

After the bounce, the general exact solution of the equations of motion is

$$\zeta(\eta, k) = \frac{\eta}{\eta^*} \left[Ce^{-i|k|(\eta-\eta^*)} + De^{+i|k|(\eta-\eta^*)} \right], \quad \eta \geq \eta^*. \quad (66)$$

The wave function of the fluctuation $\zeta(\eta, k)$ is given by the functional integral

$$\Psi[\zeta(\eta, k)] = \int [d\zeta d\Lambda] \exp(iS[\zeta, \Lambda]), \quad (67)$$

where S is given in (19). The boundary condition at $\eta \rightarrow -\infty$ is $\zeta(\eta, k) \rightarrow Ae^{-i|k|\eta}$, because the pre-big bang initial state is the standard Minkowski vacuum. We want to compute the visible primordial spectrum, so we must compute $\Psi[\zeta(\eta, k)]$ well after the bounce, at $\eta \rightarrow +\infty$. So, the value of ζ at the late-time boundary is fixed: $\zeta(\eta, k) \rightarrow \zeta(k)$. In the quadratic approximation used in (19), the wave function is Gaussian

$$\Psi[\zeta(k)] = \prod_k \exp \left[-\frac{\gamma(k)}{2} \zeta^2(k) \right]. \quad (68)$$

The power spectrum is a function of the equal time, two point correlator of ζ :

$$\langle \zeta(k) \zeta(k') \rangle = \frac{(2\pi)^3}{k^3} \delta^3(k + k') P(k) \quad (69)$$

$$\langle \zeta(k) \zeta(k') \rangle = \int [d\zeta] \zeta(k) \zeta(k') |\Psi[\zeta]|^2 / \int [d\zeta] |\Psi[\zeta]|^2 = (2\pi)^3 \delta^3(k + k') \frac{1}{2\text{Re } \gamma(k)}. \quad (70)$$

To compute $\gamma(k)$ we perform the functional integral (67). Since it is Gaussian, it reduces to

$$\Psi[\zeta(k)] = \lim_{\eta \rightarrow +\infty} \exp \left[\frac{-i}{16\pi G} \int d^3k \left(\frac{\eta}{\eta^*} \right)^2 \zeta'(\eta, k) \zeta(\eta, -k) \right]. \quad (71)$$

The boundary at $\eta \rightarrow -\infty$ has been discarded using the $+i\epsilon$ prescription to select the positive-frequency Minkowski vacuum, that is by running the contour of integration in η over Euclidean time in the far past. This prescription selects the true vacuum even in the interacting theory⁷. It is this prescription that generates a real part in γ , which is naively purely imaginary.

A major simplification occurs because we are interested in observable modes, whose wavelength is much larger than the Hubble radius immediately after the bounce: $2\pi/|k| \gg 1/H(\eta^*) = \eta^*$. In this case one can use the boundary condition at late time, $\zeta = \zeta(k)$, to approximate the post-bounce evolution of ζ as

$$\zeta(\eta, k) = \zeta(k) + E \frac{\eta^*}{\eta} + O(|k|\eta), \quad \frac{2\pi}{|k|} \gg \eta \geq \eta^*. \quad (72)$$

Before the bounce, boundary conditions at $\eta \rightarrow -\infty$ set

$$\zeta(\eta, k) = A[1 - i|k|(\eta + \eta^*)] + O(|k|^2\eta^2), \quad -\frac{2\pi}{|k|} \ll \eta \leq -\eta^*. \quad (73)$$

The matching conditions at the bounce, eqs. (63,64), are then easily solved to give

$$E = \frac{i|k|/p|\alpha|^2 + F}{1/\eta^* - F - i|k|/p|\alpha|^2} \zeta(k). \quad (74)$$

⁷See [13] for another application of this method to cosmology.

By substituting into eq. (71) we find

$$\gamma(k) = \frac{1}{8\pi G} \frac{|k|/p|\alpha|^2 - iF}{1 - \eta^*F - i|k|\eta^*/p|\alpha|^2}, \quad (75)$$

whence

$$\frac{(2\pi)^3}{k^3} P(k) = 4\pi G \frac{(1 - \eta^* \text{Re } F)^2 + (\eta^* \text{Im } F + |k|\eta^*/p|\alpha|^2)^2}{|k|/p|\alpha|^2 - \text{Im } F}. \quad (76)$$

Continuity at the bounce sets $\alpha(k) = 1$, $F(k) = 0$. In this case we get $P(k) \propto |k|^2 + \eta^*|k|^3 \sim |k|^2$, instead of the scale invariant spectrum $P(k) \propto \text{cst}$. This result agrees with ref. [11].

This means that *all* the features of the primordial spectrum are due to the unknown physics at the bounce, where the NEC is violated. This is an unavoidable [11], unpleasant feature of cyclic cosmologies, that makes the whole pre-big bang phase almost useless. Almost, but not completely. We can consider η^* as “the earliest” time and integrate out all the pre-big bang evolution, but in doing so, we would need nonlocal operators on the brane to generate a scale invariant spectrum. The long pre-big bang phase, instead, allows for one good choice of local coefficients on the brane:

$$F(k) = 0, \quad \alpha(k) = \frac{\sigma |k|^2 \eta^{*2}}{\sqrt{p}}, \quad (77)$$

where the constant σ can be related to the inflationary parameters H and ϵ introduced in section 3 by comparing eq. (76) to the standard inflationary power spectrum, $P(k) = 4\pi G H^2 / (2\pi)^3 \epsilon$, which leads to $\sigma \eta^* \sim \sqrt{\epsilon}/H$.

Another natural choice is to require that ζ is continuous at the bounce. This sets $\alpha(k) = 1$, $F(k) = (\sqrt{p}\sigma\eta^{*2}|k|)^{-1}$, with the same constant σ as before. We shall call these junction conditions the “short bounce.”

An interesting feature shared by all boundary conditions is that ζ keeps evolving well into the FRW phase, because the coefficient E of the decaying part of ζ must be much bigger than $\zeta(k)$ to create the correct scale-invariant power spectrum. For the junction conditions in eq. (77), for instance,

$$\zeta_{\text{now}} \equiv \zeta(k) \sim \sigma^{-2} (|k|\eta^*)^{-3} \zeta(\eta^*). \quad (78)$$

This evolution could be in itself a problem for the cyclic cosmology scenario. Eq. (78) shows that right after the bounce, in the radiation-dominated FRW phase, fluctuations are minuscule, compared to today. So, they can be affected by tiny inhomogeneities developing in the FRW phase, which we have ignored in our formalism. This is another potential source of fine-tuning, besides that due to our ignorance of the physics at the bounce, but we will not investigate it here.

Finally, the post-bounce evolution effectively erases any non-Gaussianity that could exist at the bounce. We shall discuss this effect in more details in section 6.

4.2 Causality Concerns

The parametrization of the unknown Planckian physics at the bounce given in eq. (19) contains instantaneous interactions at the bounce time $\pm\eta^{*8}$, so acausal propagation of signals may occur. To check if this pathology does occur, consider a plane wave propagating before the bounce along the direction x . The metric before the bounce is slowly contracting when $p \ll 1$ [see eq. (17)] so we approximate it by a Minkowski metric as in the previous subsection. The wave is then

$$\zeta(\eta, x) = \int d\omega e^{-i\omega(\eta+\eta^*-x)} f(\omega), \quad \eta < -\eta^*. \quad (79)$$

Before the bounce causality, that is propagation inside the light cone, means $\zeta(\eta, x) = 0$ for $x > \eta + \eta^* + \text{constant}$. We can set the constant to zero with a translation of the coordinate x . Thus, causality implies that $f(\omega)$ is analytic in the upper half-plane $\text{Im } \omega > 0$.

After the bounce, the general exact solution of the equations of motion is (66).

Before the bounce, for the wave given in eq. (79), $\omega = k$. We choose $\alpha(\omega)$ real at $\text{Im } \omega = 0$. In this case, the junction conditions reduce to two linear equations for C, D

$$(1 + i|k|)C + (1 - i|k|)D - F(k)(C + D) - i\frac{k}{p\alpha^2(k)}(C + D) = 0, \quad C + D = \alpha(k)f(k). \quad (80)$$

The coefficient of the right-moving wave is C for $k > 0$, and D for $k < 0$. Since $k = \omega$, this means that such coefficient always multiplies the term $\eta \exp(-i\omega\eta)$. Call it $E(\omega)$. From eq. (80) we find

$$E(\omega) = \theta(\omega)C(\omega) + \theta(-\omega)D(\omega) = \frac{i\alpha(\omega)}{2\omega} \left(1 - i\omega - F(\omega) - \frac{i\omega}{p\alpha^2(\omega)} \right) f(\omega). \quad (81)$$

Whenever $F(\omega)$, $\alpha(\omega)$ are analytic in the upper half-plane $\text{Im } \omega > 0$, and $\alpha(\omega)$ has no zeroes there, $E(\omega)$ is analytic in the same region, and no propagation outside the light cone occurs after the bounce. This happens for the matching conditions (77). On the other hand, the “short” bounce condition, as well as most other acceptable ones, is non analytic: $F(\omega) = 1/\sigma\eta^{*2}|\omega|$ has a cut along the ω imaginary axis.

The bounce explicitly breaks Lorentz invariance, so faster-than light propagation does not necessarily imply causality violations. After all, one needs a source before the big bang to get a signal that can propagate faster than light *after* the big bang. Yet, the pathology is troublesome. So, in the next subsection, we shall restrict our attention to the junction conditions (77) and see whether they satisfy our naturalness criterion. Afterward, in section 4.4, we shall return to studying more general boundary conditions.

⁸Identification of these two times implies an instantaneous bounce. Since we will find acausal signals propagating with arbitrarily high speed, this simplification will not affect our conclusions.

4.3 Naturalness Concerns I

The quadratic junction conditions given in subsection 4.1 need to be covariantized with respect to 3-d coordinate transformations. This is particularly simple for the conditions (77). It suffices to notice that at linear order $\partial^2\zeta \sim \mathcal{R}$. Then the action contains the term

$$S_3 = \frac{\sigma\eta^{*2}}{16\pi G\sqrt{p}} \int d^3x \sqrt{h} \mathcal{R} \Lambda \Big|_{-\eta^*}. \quad (82)$$

By expanding this equation we get a coupling $S_3^{\zeta^3\Lambda} \sim \zeta^2 \partial^2 \zeta \Lambda$. The equal-time ζ propagator is now $\propto |k|^{-1}$ so $S_3^{\zeta^3\Lambda}$ produces a self-energy diagram as in fig. 3, which gives a one-loop correction of the form (recall that $a(\eta^*) = 1$ here)

$$S_3^{\zeta^2} \sim \frac{\sigma\eta^{*2}M^4}{4\pi^2\sqrt{p}} \int d^3x \zeta \Lambda \Big|_{-\eta^*}. \quad (83)$$

This correction destroys the scale invariance of the spectrum for all momenta $|k| < O(\sqrt{GM^4})$.

Again, a scale invariant spectrum turns out to be possible by fine-tuning the operators inserted on the brane. Since they parametrize our ignorance of the high-energy physics occurring at the bounce, this means that a scale invariant spectrum depends on a very special completion of our EFT instead of being a robust, generic feature.

Our analysis requires that we do not know in detail the history of the bounce. If we know what happens at the bounce, then we cannot invoke a genericity argument. To appreciate this point, imagine that we ignore the whole history of the universe before the bounce, i.e. we integrate out $\zeta(\eta)$ for $\eta < -\eta^*$ and we substitute for it the boundary term

$$S_3 = \frac{1}{16\pi G} \int \frac{d^3k}{(2\pi)^3} i|k| |\zeta(\eta, k)|^2 \Big|_{-\eta^*}. \quad (84)$$

A generic covariantization of this term would produce a self-energy correction resulting in a “mass” term $\sim M^3\zeta^2$ that changes the spectrum for all momenta $|k| \lesssim O(GM^3)$. In this case though, we *do* know the previous history of the universe. It is a long slowly contracting phase, in which the power spectrum $1/|k|$ is due to the masslessness of ζ , which is guaranteed by it being part of a gauge field: the metric. Essentially the same argument tells us that the scale-invariant power spectrum of inflation *is* natural.

4.4 Naturalness Concerns II

The function $F(\partial)$ that we introduced in eq. (19) parametrizes how the superhorizon modes get modified during the bounce. As we noted earlier in this section, in the case of a short bounce this function must be nonlocal to reproduce scale invariance at late times:

$$F(\partial) = \frac{1}{\sqrt{p}\sigma\eta^{*2}\sqrt{-\partial^2}}. \quad (85)$$

The nonlocality of this interaction on the S-brane complicates the study of its stability under radiative corrections.

Our strategy is to rewrite the part of the action containing $F(\partial)$ as a fully local action, by introducing auxiliary fields localized on the S-brane. Covariantization of this local action introduces then certain universal interactions of ζ with the additional fields, which allow us to study the radiative stability of this system in the same way as in the previous section.

To introduce the technique, consider a three dimensional scalar with action:

$$S[\phi] = \int d^3x \phi \frac{1}{\sqrt{-\partial^2}} \phi. \quad (86)$$

To render this action local, introduce an additional, massless field φ that couples to ϕ as follows:

$$S[\phi, \varphi] = \int d^3x \left[\lambda \phi \varphi^2 + \varphi \partial^2 \varphi \right]. \quad (87)$$

In terms of the variables ϕ, φ , this action is fully local. The field φ appears quadratically in it, so we can integrate φ out exactly. After integration, we are left with an effective action for ϕ only, which possesses a nonlocal propagator. A calculation of the diagram in figure 5 gives:

$$S_{eff}[\phi] = \frac{1}{4\pi} \int d^3x \phi \frac{\lambda^2}{\sqrt{-\partial^2}} \phi + \dots, \quad (88)$$

The ellipses denote terms of order ϕ^3 or higher, that we do not need for computing the power spectrum. The quadratic term is indeed the nonlocal action of eq. (86). We note that this mechanism depends crucially on the masslessness of the scalar φ . If radiative corrections induce a mass m for φ , the form of the action eq. (86) is spoiled for all momenta $|k| \lesssim m$.

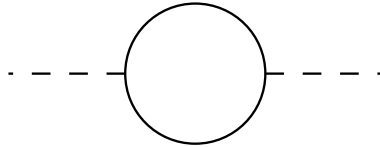


Figure 5: The one-loop, UV finite diagram giving rise to the effective action in equation (88). The solid line denotes φ , whereas dashed lines denote ϕ .

In our case, the fundamental scalar is the ζ -field. We cannot exactly use the above mechanism with $\phi \rightarrow \zeta$, because the action in eq. (87) cannot be covariantized consistently. Indeed, the interaction term $\zeta \varphi^2$ is uniquely covariantized by the local expression $\sqrt{h} \varphi^2$. At zeroth order in the metric fluctuation however, this term gives a mass to φ , which makes the one-loop induced term regular in the infrared, instead of being divergent as $|k|^{-1}$.

For this reason we must resort to a more complicated system to generate an action of the form (86) for ζ . The following covariant action satisfies our requirements:

$$S[\zeta, \varphi, \psi] = \int d^3x \sqrt{h} \left[\alpha \mathcal{R} \varphi + \beta \bar{\psi} (\not{D} \Delta \varphi) \psi + \bar{\psi} \Delta \psi \right]. \quad (89)$$

Here φ is a scalar, ψ is a spinor, and Δ is the covariant Laplacian. Notice that these fields have non-minimal kinetic terms. To see how the effective action for ζ looks like we proceed in two steps, by first integrating out ψ and next φ . Integrating out ψ at one-loop, as shown in figure 6, we arrive at an effective action for φ and ζ that to quadratic order reads:

$$S[\zeta, \varphi] = \int d^3x \left[\alpha \partial^2 \zeta \varphi + \beta^2 \varphi (-\partial^2)^{5/2} \varphi \right]. \quad (90)$$

Next, integrating out φ the effective quadratic action for ζ reduces to:

$$S_{eff}[\zeta] = \int d^3x \left[\zeta \frac{\alpha^2}{\beta^2 \sqrt{-\partial^2}} \zeta \right] \quad (91)$$

By appropriately choosing α and β , we can make $\alpha^2/\beta^2 = (\sqrt{p}\sigma\eta^{*2})^{-1}$, so that this interaction at the S-brane is indeed $\zeta F(\partial)\zeta$.

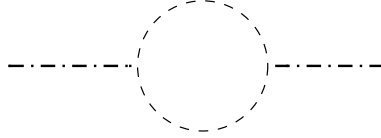


Figure 6: The one-loop diagram giving rise to the effective action in equation (91). Dashed lines represent ψ , whereas dashed-dotted lines represent φ .

The Lagrangian in (89) is our starting point for investigating if the induced action in (91) is stable under radiative corrections. The interactions in (89) come from the covariantization of the action, with ζ coupling universally to the other fields in the action. Now, we must check if any of these interactions feed down to dangerous relevant operators. For us a “dangerous operator” means an operator that spoils the scale invariance of the spectrum after the bounce, such as a large mass term for ψ or φ . In principle, a mass smaller than the IR-cutoff, $aH|_{now}$, is allowed, since we cannot probe the spectrum below this scale. After all, the function $F(k) \sim |k|^{-1}$ should be regulated by that IR cutoff in order not to generate infinite backreaction at zero momentum. The point is that we will show that the *natural* induced mass is much larger than the IR cutoff.

For example, we see that the covariant kinetic term for the ψ -field in eq. (89) contains terms like:

$$\Delta S^{\zeta\bar{\psi}\psi} \sim \int d^3x \bar{\psi} \partial_i \zeta \partial^i \psi. \quad (92)$$

This interaction gives rise to a mass term by the one-loop diagram shown in figure 7. Cutting off the momentum integral at the scale M , we obtain:

$$\Delta S^{\bar{\psi}\psi} \sim \frac{M^4}{M_p^2} \int d^3x \bar{\psi} \psi. \quad (93)$$

This relevant interaction dominates the original kinetic term for ψ at momentum scales $|k| \lesssim \frac{M^2}{M_p}$.



Figure 7: The one-loop diagram giving rise to the effective action in equation (91). Dashed lines represent ψ and the solid line represents ζ .

More importantly, the one-loop induced operator in eq. (90) is modified to:

$$\Delta S^{\varphi^2} \sim \beta^2 \frac{M_P^4}{M^5} \int d^3x \varphi (\partial^2)^3 \varphi. \quad (94)$$

Now, by integrating out the field φ we obtain the following operator:

$$\Delta S^{\zeta^2} \sim \frac{1}{\sqrt{p\sigma\eta^{*2}}} \frac{M^5}{M_P^4} \int d^3x \zeta \frac{1}{\partial^2} \zeta. \quad (95)$$

This term is clearly IR dominant over the original term in eq. (91), for all momenta $|k| \lesssim M(M/M_p)^4$ and it produces an unacceptably red spectrum after the bounce over the whole range of scales relevant for observation. To avoid the appearance of this term, we must tune the mass term of the fermion in eq. (93) to the much smaller value (by about thirty orders of magnitude!) discussed above, by adding a counterterm in the bare Lagrangian. This is the exact analogue of the mass instability of light fundamental scalars in the standard model. Our analysis here clearly indicates that the choice for the function $F(\partial)$ in eq. (85) is fine-tuned in an effective field theory perspective, and that only a very specific UV-completion of the short bounce can guarantee scale invariance at late times.

One can show that similar considerations apply to other choices of boundary non-local operators that reproduce the correct power-spectrum when inserted in eq. (76). By an appropriate choice of auxiliary fields one can make these operators local. This always requires some of the auxiliary fields to have an unnaturally small value for the coefficient of an unprotected operator, that receives correction due to its universal interactions with the metric.

5 Non-Gaussianities in Inflation

5.1 Minimally Coupled Scalar in de Sitter Space

A minimally coupled scalar models tensor fluctuations in the CMB, so it has an intrinsic interest besides offering a simple example of our method. The scalar action $S_4 + S_3$, given in eq. (1), is only the quadratic part of an action that can contain nonlinear terms both in the 4-d bulk and in the 3-d boundary. We shall consider here the effect of adding a boundary interaction

$$\Delta S_3^{\chi^3} = \int d^3x a^3(\eta^*) \lambda \chi^3, \quad (96)$$

while keeping the bulk action S_4 quadratic. We could compute the effect of this term on late-time correlators in a Hamiltonian formalism, as in subsection 2.3, but we will use instead a Lagrangian approach, which is simpler for taking into account back-reaction effects and for application to cyclic cosmology. In this case, the effect at late time $\eta \rightarrow 0^-$, is [cfr. eq. (27)]

$$\Psi[\chi_{now}] = \int [d\chi..] e^{i(S_4 + S_3 + \Delta S_3^{\chi^3})}. \quad (97)$$

The boundary condition at late time is $\chi(0, x) = \chi_{now}(x)$ while at early time is obtained by making the action $S_4 + S_3 + \Delta S_3^{\chi^3}$ stationary w.r.t. free variations of χ , including those that do not vanish on the boundary. The resulting initial condition, written for the Fourier components $\chi(0, k)$, is

$$\chi'(\eta^*, k) + \tilde{\kappa}(k)\chi(\eta^*, k) + 3\lambda a(\eta^*) \int \frac{d^3 l}{(2\pi)^3} \chi(\eta^*, l) \chi(\eta^*, k - l) = 0. \quad (98)$$

In the bulk, χ obeys a free equation of motion, whose general solution is

$$\chi(\eta, k) = A(1 - i|k|\eta) \exp(-i|k|\eta) + B(1 + i|k|\eta) \exp(i|k|\eta). \quad (99)$$

At tree level, the wave function (96) is $\Psi[\chi] = \exp(iS_4 + iS_3 + i\Delta S_3^{\chi^3})$, computed on shell. We only need quadratic and cubic terms in the action, so we can expand the solution of the equations of motion as $\chi(\eta, k) = \chi_0(\eta, k) + \lambda\chi_1(\eta, k)$, and keep only terms at most of linear order in λ . The coefficients $\chi_0(\eta, k)$, $\chi_1(\eta, k)$ obey the boundary conditions

$$\chi_0(0, k) = \chi(k), \quad \chi_1(0, k) = 0, \quad \chi'_0(\eta^*, k) + \tilde{\kappa}(k)\chi_0(\eta^*, k) = 0. \quad (100)$$

We need not write the boundary condition for $\chi_1(\eta^*, k)$ because of the following reason. On shell and to linear order in λ , we can integrate by part the free action and use the free bulk equations of motion to arrive at

$$\begin{aligned} S_4 + S_3 + \Delta S_3^{\chi^3} &= \lim_{\eta \rightarrow 0^-} \left\{ -\frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} a^2(\eta) \chi'_0(\eta, k) [\chi_0(\eta, -k) + 2\lambda\chi_1(\eta, -k)] + \right. \\ &\quad \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} a^2(\eta^*) [\chi'_0(\eta^*, k) + \tilde{\kappa}(k)\chi_0(\eta^*, k)] [\chi_0(\eta^*, -k) + 2\lambda\chi_1(\eta^*, -k)] + \\ &\quad \left. \int d^3 x a^3(\eta^*) \lambda \chi_0^3(\eta^*, x) \right\}. \end{aligned} \quad (101)$$

Boundary conditions (100) now make all terms within brackets vanish, except the first that reduces to $\chi(-k)$. Therefore, the action reduces to the standard quadratic term, which gives rise to Gaussian fluctuations, plus a cubic term obtained simply by writing the free field χ at η^* in terms of its late-time value $\chi(k)$.

$$S_4 + S_3 + \Delta S_3^{\chi^3} = \int \frac{d^3 k}{(2\pi)^3} i \frac{|k|^3}{H^2} |\chi(k)|^2 + \Delta S_3^{\chi^3} [\chi_0^3(\eta^*, k)]. \quad (102)$$

In this formula, we have dropped a real divergent term, which does not contribute to expectation values, since it disappears in taking the square norm of the wave function. Using eq. (99) and the boundary conditions (100), the cubic term finally reads

$$\Delta S_3^{\chi^3}[\chi_0^3(\eta^*, k)] = \lambda \int \frac{d^9 k}{(2\pi)^9} a^3(\eta^*) (2\pi)^3 \delta^3(k_1 + k_2 + k_3) \prod_{j=1}^3 \chi(k_j) (1 - i|k_j|\eta^*) \exp(i|k_j|\eta^*). \quad (103)$$

The cubic non-Gaussianity is

$$\langle \chi(k_1) \chi(k_2) \chi(k_3) \rangle = \frac{\int [d\chi] \chi(k_1) \chi(k_2) \chi(k_3) |\Psi[\chi(k)]|^2}{\int [d\chi(k)] |\Psi[\chi(k)]|^2}. \quad (104)$$

To first order in λ this integral is

$$\langle \chi(k_1) \chi(k_2) \chi(k_3) \rangle = 6(2\pi)^3 \delta^3(k_1 + k_2 + k_3) \tilde{\lambda}(k_1, k_2, k_3) \prod_{j=1}^3 \frac{H^2}{2|k_j|^3} \quad (105)$$

$$\tilde{\lambda}(k_1, k_2, k_3) = -2a^3(\eta^*) \text{Im} \left[\prod_{j=1}^3 (1 - i|k_j|\eta^*) \exp(i|k_j|\eta^*) \right]. \quad (106)$$

Notice that the oscillating exponent in $\tilde{\lambda}$ makes it vanish for momenta $|k| \gg 1/|\eta^*|$, i.e. for physical momenta $|k|/a \gg H$. So, the effect of the boundary term is significant only for momenta that crossed the horizon *before* the time η^* . Physically, this effect can be understood by interpreting the boundary term ΔS^{χ^3} as a “summary” of cosmic evolution prior to η^* -such as phase transitions, decoupling of heavy particles etc.. In this picture, any effect of this early history is washed away by inflation in all modes that keep evolving after η^* . This cutoff at η^* is another manifestation of the key property of inflating backgrounds, namely that they dilute away any primordial inhomogeneity. It also makes it difficult to detect such a non-Gaussianity unless $Ha(\eta^*)/a(\eta_{\text{now}}) > H_{\text{now}}$.

For momenta $|k| \ll 1/|\eta^*|$ and λ real, eq. (106) becomes

$$\langle \chi(k_1) \chi(k_2) \chi(k_3) \rangle = -(2\pi)^3 \delta^3(k_1 + k_2 + k_3) \frac{H^3 \lambda}{2} \sum_{l>m} |k_l|^{-3} |k_m|^{-3}. \quad (107)$$

5.1.1 Estimating the Back-Reaction

The coupling $\lambda \chi^3$ has dimension 3 so it is marginal in the boundary EFT. Loops can only renormalize it logarithmically, so it can be naturally small. At $O(\lambda^2)$, there exists a potential contribution to the boundary “mass” term $\int d^3 x a^3 \chi^2$. On dimensional grounds, we can easily estimate its coefficient as $O(\lambda^2 M)$. So, as long as

$$\lambda^2 \lesssim \epsilon H/M, \quad (108)$$

the induced “mass” term is not in conflict with experimental bounds.

5.2 Scalar Fluctuations

Scalar fluctuations ζ have model independent non-Gaussianities [13, 21] of the form

$$\langle \zeta^3 \rangle = (2\pi)^3 \delta^3(k_1 + k_2 + k_3) \frac{64\pi^2 G^2 H^4}{\epsilon} F(k_1, k_2, k_3), \quad (109)$$

where F is a homogeneous function of degree -6 in the momenta. To compare with the results of the previous subsection, it is convenient to express the three point function in terms of the canonically normalized field v defined in eq. (32). Then the size of $\langle v^3 \rangle$ is $O(\sqrt{8\pi G\epsilon} H^4 k^{-6})$. Eventual non-Gaussianities in the initial conditions are instead $O(H^3 \lambda k^{-6})$. Clearly, for λ large enough, the boundary non-Gaussianity can be dominant for momenta $|k| < 1/|\eta^*|$. We must only check that the new cubic interactions do not introduce unacceptably large boundary “mass” renormalization. We already saw that this condition gives the bound $\lambda \lesssim \sqrt{\epsilon H/M}$. Another bound comes from the mixed term due to the universal bulk cubic interaction [13] $\sim \mu v \partial v \partial v$, where $\mu = O(\sqrt{8\pi G\epsilon})$. It is a dimension-5 operator that can induce at a boundary term $O(\mu\lambda)$ via the self-energy diagram shown in fig. 8. On dimensional grounds the induced boundary

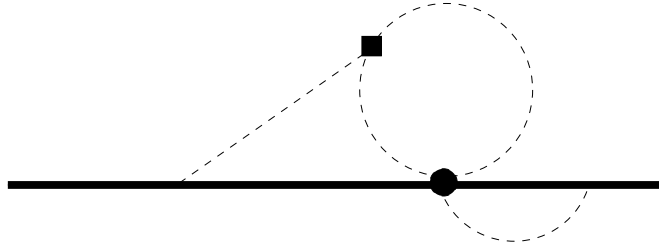


Figure 8: A mixed bulk-boundary interaction may generate a boundary mass term. The black square denotes the bulk interaction $\mu v \partial v \partial v$, the black circle denotes the boundary interaction λv^3 .

“mass” term is at most $O(\lambda \mu M^2) \int d^3 x a^3 \chi^2$. This gives another bound on λ :

$$\lambda \lesssim \sqrt{\epsilon} \frac{H}{M} \frac{M_p}{M}, \quad M_p \equiv (8\pi G)^{-1/2}. \quad (110)$$

For $\epsilon \sim 10^{-2}$, $M \sim 10^{16}$ GeV and $H \sim 10^{14}$ GeV this bound is even weaker than (108).

The ratio of the non-Gaussianity induced by boundary terms over the universal bulk term is $\lambda M_p / \sqrt{\epsilon} H$. For the extremal value $\lambda \sim \sqrt{\epsilon H/M}$, it becomes $M_p / \sqrt{HM} = O(10^4)$. So, there is definitely room for an observable signal here! A word of caution is necessary, though. First of all, in order to see a clear signal we need that inflation does not last too long. Otherwise, thanks to the cutoff at $|k| \sim 1/|\eta^*|$, the initial non-Gaussianity would only affect unobservable super-horizon fluctuations. Moreover, in deriving the bound $\lambda \lesssim \sqrt{\epsilon H/M}$ we only demanded that the induced change in the power spectrum is less than ϵ . More stringent bounds on the power spectrum give a stronger bound on λ [15].

6 Non-Gaussianities in Cyclic Cosmologies

6.1 Damping Effects

As we mentioned earlier, in cyclic cosmologies, a scale invariant power spectrum for scalar fluctuations needs significant evolution after the bounce, well into the (radiation dominated) FRW phase. For the junction conditions (77), for instance, $\zeta_{now} \equiv \zeta(k) = \sigma^{-2}(|k|\eta^*)^{-3}\zeta(\eta^*) = \sigma^{-1}(|k|\eta^*)^{-1}\zeta(-\eta^*)$. Suppose now that the wave function of the fluctuations just before the bounce contains a non-Gaussianity

$$\Psi[\zeta_-] = \prod_k \exp \left[-\frac{|k|}{16\pi G} |\zeta_-(k)|^2 + \lambda(k_1, k_2, k_3) \zeta_-(k_3) \zeta_-(k_3) \zeta_-(k_3) \right], \quad \zeta_- \equiv \zeta(-\eta^*). \quad (111)$$

The three-point function can be computed as we did for inflation:

$$\langle \zeta_-^3 \rangle = \frac{\int [d\zeta] \zeta^3 |\Psi[\zeta]|^2}{\int [d\zeta] |\Psi[\zeta]|^2} \sim \lambda \frac{(16\pi G)^3}{|k|^3}. \quad (112)$$

Assume for sake of example that bulk non-Gaussianities are negligible; then the late-time form of Ψ is obtained by evolving ζ with its quadratic equations of motion, and by expressing ζ_- in terms of ζ_{now} : $\zeta_- = \sigma|k|\eta^*\zeta_{now}$

$$\Psi[\zeta_{now}] = \prod_k \exp \left[-\frac{\sigma^2 \eta^{*2} |k|^3}{16\pi G} |\zeta_{now}(k)|^2 + \tilde{\lambda}(k_1, k_2, k_3) \zeta_{now}(k_3) \zeta_{now}(k_3) \zeta_{now}(k_3) \right] \quad (113)$$

$$\tilde{\lambda}(k_1, k_2, k_3) = \sigma^3 \eta^{*3} |k_1| |k_2| |k_3| \lambda(k, k_2, k_3). \quad (114)$$

Finally, the magnitude of $\langle \zeta_{now}^3 \rangle$ is

$$\langle \zeta_{now}^3 \rangle / (\langle \zeta_{now}^2 \rangle)^{3/2} \sim (\sigma \eta^* |k|)^3 \langle \zeta_-^3 \rangle / (\langle \zeta_-^2 \rangle)^{3/2} \ll \langle \zeta_-^3 \rangle / (\langle \zeta_-^2 \rangle)^{3/2}. \quad (115)$$

So, any pre-bounce non-Gaussianity is damped by the extremely small factor $(\sigma \eta^* |k|)^3$. Similar estimates hold for other choices of junction conditions.

Evidently, only very dramatic non-Gaussianities have a chance to survive the smoothing effect of the bounce. In the next subsection we investigate whether this may happen.

6.2 Non-Linear Interactions in the Slow-Contraction Epoch

There is one last subtlety about the size of non-Gaussianities in the cyclic models we are considering that needs been addressed. It relates to the consistency of the perturbative expansion in this backgrounds. Looking at the scalar potential (17), one may worry that for small p the nonlinearities of the theory may become important, and the model may become strongly coupled. Indeed, if we consider the action to non-linear order, we may worry about terms containing self-interactions of the scalar, since on our background, for instance, $V'''(\phi) \sim 1/\sqrt{p}$.

Consider the situation in the harmonic (de Donder) gauge, in which the kinetic term of the metric reduces to $\hat{h}_\mu^\nu \square \hat{h}_\nu^\mu$, where $\hat{h}_\mu^\nu = h_\mu^\nu - 1/2 \delta_\mu^\nu h^\lambda_\lambda$, and *every* component of the metric propagates. In this gauge, all interactions that involve the background metric are necessarily proportional to positive powers of H/M_p or ϕ'/M_p , since we are not solving for some of the fields in terms of other fields using the constraints. The only potentially large interaction is the cubic term in the expansion for the scalar field potential. Using the background solution, eq. (17), this term reads

$$\Delta S^{\phi^3} \simeq \int d^4x \frac{1}{3} \sqrt{\frac{8\pi G}{3p}} \frac{1}{\eta^2} \varphi^3, \quad (116)$$

The crucial fact is that this term is time-dependent. This means that we can trust perturbation theory *before* some critical time η_c that will be related to the value of p , but not afterwards⁹. In other words, we cannot follow the contracting solution arbitrarily close to the singularity. This means that, if we want to use perturbation theory, we are forced to place our S-brane at an earlier time than η_c .

An easy way to get an estimate for η_c is to consider the quartic coupling, $V''''(\phi)\varphi^4$. This is a dimension four operator, and its dimensionless coupling is $(pM_p^2\eta^2)^{-1}$. By requiring that this is smaller than $O(1)$, we arrive at the estimate $\eta_c \approx (M_p\sqrt{p})^{-1}$. Since we know that $\eta^*\sigma = \sqrt{\epsilon}H^{-1}$, and we need $\eta^* > \eta_c$ our estimate of η^c translates into $p > (H/\sqrt{\epsilon}M_p)^2\sigma^2 \approx 10^{-5}\sigma^2$. This should be combined with the requirement that the initial time is not outside of the effective field theory regime, $\eta^* > 1/M$.

7 Conclusions

We presented a general method to describe unknown “dark” periods in cosmology, where high energy/Planckian curvature effects become important. These are, among others, the Planckian era at the beginning of inflation, and the singularity (bounce) in cyclic cosmologies. The unknown cosmic evolution during such times is replaced by a spacelike brane which can be endowed with extra degrees of freedom localized on it. These extra degrees of freedom allow for a general parametrization of the unknown UV physics by local operators of ever increasing dimension.

There is an infinite number of irrelevant operators, which parametrize the effect of the dark period and its unknown UV physics, as well as a finite number of relevant operators. On the boundary, relevant operators have dimension strictly less than three.

If the coefficients of the relevant operators are too small, they may be unstable against changes in the UV physics (or, equivalently, the details of the cosmic evolution during the

⁹Recall that in the contracting solution we are considering, time runs from $\eta = -\infty$ to $\eta = 0$, at which the space-time is singular.

dark period). The EFT parametrizes our ignorance of the dark period, so small coefficients of relevant operators signal a non-generic evolution, which is fine-tuned unless we have some additional knowledge of the dark period itself. From the point of view of the EFT then, such parameters are unnatural.

We applied this naturalness requirement to constrain possible deviations of the initial state of inflation from the BD vacuum. We found that the generic signal of any such modification in the power spectrum of primordial fluctuations is a universal boundary “mass” term, no matter what is the origin of the UV modification. From this point of view, the power spectrum is not well suited to detect a clear, distinctive signal of new UV physics. For instance, trans-Planckian modifications to dispersion relations and late-time phase transitions during inflation would give an essentially identical signal.

The naturalness requirement is quite powerful in constraining cyclic cosmologies. From this point of view, they are all fine-tuned. Many of them also allow for faster-than light propagation of signals and may be unstable against small inhomogeneities during the radiation-dominated phase of cosmic evolution.

We also studied other signals of changes in initial conditions, besides those affecting the spectrum. We found that a short inflation still allows some initial non-Gaussianities to survive and be large enough for detection. Unlike the case of changes in the power spectrum, these non-Gaussianities can be large without generically inducing a large distortion of the power spectrum.

Finally, we studied non-Gaussian features in cyclic cosmologies. We found that these cosmologies generically damp all pre-bounce non-Gaussianities. We also showed that the request of no exceedingly large non-linear interactions in the pre-bounce, slow contraction epoch, demands that the S-brane cuts off cosmic evolution at a time parametrically larger than the Planck time.

Our method extends the formalism set up in [7] to situations where cosmic evolution is known after *and before* a “dark” era, but not during it. The dark era is excised and substituted with an S-brane. It is fascinating to speculate that this method may shed new light on other cases where matter passes through (or ends on) a region that can be traded for an S-brane. The most interesting case is of course the singularity inside a non-extremal black hole, which is conjectured to be generically spacelike and of BKL [22] type.

We want to end on a constructive note, by pointing out that naturalness could also become a guiding principle in the search for a UV completion of cosmic evolution near singularities, much in the same spirit as naturalness has guided particle physics in the search for a UV completion of the standard model.

Acknowledgments

We would like to thank K. Schalm and J.P. van der Schaar for discussions and comments. F. Nitti is supported by an NYU McArthur Fellowship. M.P. is supported in part by NSF grant PHY-0245068, and in part by a Marie Curie Chair, contract MEXC-CT-2003-509748 (SAG@SNS). J.-W.R. is supported by an NYU Henry M. McCracken Fellowship.

References

- [1] C. L. Bennett *et al.*, *Astrophys. J. Suppl.* **148**, 1 (2003) [arXiv:astro-ph/0302207].
- [2] N. D. Birrell and P. C. W. Davies, *Quantum Fields In Curved Space*, Cambridge Univ. Press (1982).
- [3] J. Martin and R. H. Brandenberger, arXiv:astro-ph/0012031.
- [4] N. Kaloper, M. Kleban, A. E. Lawrence and S. Shenker, *Phys. Rev. D* **66**, 123510 (2002) [arXiv:hep-th/0201158]; N. Kaloper, M. Kleban, A. Lawrence, S. Shenker and L. Susskind, *JHEP* **0211**, 037 (2002) [arXiv:hep-th/0209231].
- [5] C. P. Burgess, J. M. Cline, F. Lemieux and R. Holman, *JHEP* **0302**, 048 (2003) [arXiv:hep-th/0210233]; C. P. Burgess, J. M. Cline and R. Holman, *JCAP* **0310**, 004 (2003) [arXiv:hep-th/0306079]; C. P. Burgess, J. Cline, F. Lemieux and R. Holman, arXiv:astro-ph/0306236.
- [6] N. A. Chernikov and E. A. Tagirov, *Annales Poincaré Phys. Theor. A* **9** 109 (1968); E. A. Tagirov, *Annals Phys.* **76**, 561 (1973); E. Mottola, *Phys. Rev. D* **31**, 754 (1985); B. Allen, *Phys. Rev. D* **32**, 3136 (1985); J. G  h  niau and C. Schomblond, *Bull. Class. Sci. Acad. Roy. Belg.* **54**, 1147 (1968); C. Schomblond and P. Spindel, *Annales Poincar   Phys. Theor. A* **25**, 67 (1976).
- [7] K. Schalm, G. Shiu and J. P. van der Schaar, arXiv:hep-th/0401164.
- [8] B. R. Greene, K. Schalm, G. Shiu and J. P. van der Schaar, arXiv:hep-th/0411217.
- [9] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, *Phys. Rev. D* **64**, 123522 (2001) [arXiv:hep-th/0103239]; P. J. Steinhardt and N. Turok, *Phys. Rev. D* **65**, 126003 (2002) [arXiv:hep-th/0111098].
- [10] M. Gasperini and G. Veneziano, *Phys. Rept.* **373**, 1 (2003) [arXiv:hep-th/0207130].
- [11] P. Creminelli, A. Nicolis and M. Zaldarriaga, arXiv:hep-th/0411270.

- [12] V. Bozza and G. Veneziano, arXiv:hep-th/0502047.
- [13] J. Maldacena, JHEP **0305**, 013 (2003) [arXiv:astro-ph/0210603].
- [14] M. Porrati, Phys. Lett. B **596**, 306 (2004) [arXiv:hep-th/0402038].
- [15] M. Porrati, arXiv:hep-th/0409210.
- [16] J. M. Bardeen, Phys. Rev. D **22** (1980) 1882.
- [17] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rept. **215**, 203 (1992).
- [18] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, Phys. Rev. D **66**, 046005 (2002) [arXiv:hep-th/0109050].
- [19] F. Larsen and R. McNees, JHEP **0307**, 051 (2003) [arXiv:hep-th/0307026]; JHEP **0407**, 062 (2004) [arXiv:hep-th/0402050].
- [20] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).
- [21] N. Bartolo, E. Komatsu, S. Matarrese and A. Riotto, Phys. Rept. **402**, 103 (2004) [arXiv:astro-ph/0406398]; V. Acquaviva, N. Bartolo, S. Matarrese and A. Riotto, Nucl. Phys. B **667**, 119 (2003) [arXiv:astro-ph/0209156].
- [22] V. A. Belinsky, I. M. Khalatnikov and E. M. Lifshitz, Adv. Phys. **19**, 525 (1970); Sov. Phys. JETP **35**, 838 (1972); Adv. Phys. **31**, 639 (1982).